

C^∞ REGULARITY OF CERTAIN THIN FREE BOUNDARIES

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ABSTRACT. We continue our study of the free boundary regularity in the thin one-phase problem and show that $C^{2,\alpha}$ free boundaries are smooth.

1. INTRODUCTION

In this paper we investigate C^∞ regularity of the free boundary in the *thin one-phase* problem. In general a thin free boundary refers to a problem in which the free boundary is expected to have codimension 2.

We consider the thin one-phase problem (or thin Bernoulli problem) which consists in finding a non-negative function

$$u : \overline{B_1} \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}, \quad u \in C(\overline{B_1})$$

with prescribed values $u = \varphi \geq 0$ on ∂B_1 , such that u satisfies

$$\{u = 0\} \subset \{x_{n+1} = 0\},$$

and

$$(1.1) \quad \begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ \frac{\partial u}{\partial \sqrt{t}} = 1 & \text{on } \Gamma := \partial_{\mathbb{R}^n} \{u = 0\} \subset \{x_{n+1} = 0\}. \end{cases}$$

We used the notation

$$\frac{\partial u}{\partial \sqrt{t}}(Z) := \lim_{t \rightarrow 0^+} \frac{u(z + t\nu, 0)}{t^{1/2}}, \quad Z = (z, 0) \in \Gamma,$$

and ν denotes the outward normal to the free boundary Γ in \mathbb{R}^n . There is an energy functional associated to this problem,

$$(1.2) \quad E(u) := \int |\nabla u|^2 dX + \frac{\pi}{2} \mathcal{H}^n(\{u > 0\} \cap \{x_{n+1} = 0\}),$$

and solutions to (1.1) are critical points for E .

To fix ideas we explain the situation in the simplest case $n = 1$. Typically u vanishes continuously on a number of segments on $\{x_2 = 0\}$ and u is positive harmonic on the two dimensional disk away from these segments. In this case the free boundary Γ consists of the endpoints of these horizontal segments. A harmonic function grows on the x_1 -axis as $a d^{1/2} + o(d^{1/2})$ away from its vanishing segments, for some constant a , where d represents the distance to the zero set. The free boundary condition above requires that the constant a must be 1 for all endpoints. It can be understood as a Neumann type condition which determines the set $\{u = 0\}$.

Key words and phrases. One-phase free boundary problem; Schauder estimates.

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The thin one-phase free boundary problem was first considered by Caffarelli, Roquejoffre and Sire [CRS] as a model of a one-phase Bernoulli type free boundary problem in the context of the fractional Laplacian. It appears in flame propagation when turbulence or long-range interactions are present. When $n = 2$, the problem (1.1) is related to models involving traveling wave solutions for planar cracks. In this setting $\{u = 0\}$ represents the location of the crack in a 3D material and the free boundary Γ is one-dimensional and represents the edge of the crack. For further information on this model see [CRS] and the references therein.

The study of the regularity of thin one-phase free boundaries was initiated in [DR], where it was shown that “flat” free boundaries are $C^{1,\alpha}$. In [DS1], [DS2] we continued investigating this regularity issue. These results parallel the regularity theory for the free boundary in the classical one-phase problem and in the theory of minimal surfaces. We showed that Lipschitz free boundaries are of class $C^{2,\alpha}$ and local minimizers of E have $C^{2,\alpha}$ free boundary except possibly for a small singular set of Hausdorff dimension $n - 3$. In the current paper we address the issue of higher regularity of the free boundary. We prove that $C^{2,\alpha}$ free boundaries are in fact smooth.

Theorem 1.1. *Assume u satisfies (1.1) and $\Gamma \in C^{2,\alpha}$. Then $\Gamma \in C^\infty$.*

The techniques developed in this paper are quite general and can be used to investigate the higher regularity of other thin free boundaries. One example of thin free boundary arises in the so-called *thin obstacle problem* also known as the *Signorini problem* (see for example [ACS, CSS, GP]).

The main difficulty in the thin one-phase problem occurs near the free boundary where all derivatives of u blow up and the problem becomes degenerate. We discuss briefly the free boundary regularity in the case of the classical Bernoulli problem ([AC, C1, C2]):

$$(1.3) \quad \begin{cases} \Delta u = 0 & \text{in } \{u > 0\}, \\ |\nabla u| = 1 & \text{on } \Gamma := \partial\{u > 0\}. \end{cases}$$

The analyticity of $C^{1,\alpha}$ free boundaries Γ was obtained by Kinderlehrer, Nirenberg and Spruck in [KNS]. They used the hodograph transform to reduce the problem to a nonlinear Neumann problem with fixed boundary. We sketch below an equivalent argument to prove higher regularity of Γ in (1.3). It avoids the hodograph transformation and it makes use of Schauder estimates for both a Dirichlet and a Neumann problem. We will follow this strategy also in the proof of our main result Theorem 1.1.

Assume that $\Gamma \in C^{k+2,\alpha}$ for some $k \geq 0$. Then by Schauder estimates for the Dirichlet problem in the set $\{u > 0\}$ we find

$$(1.4) \quad \Gamma \in C^{k+2,\alpha} \quad \Rightarrow \quad u \in C^{k+2,\alpha}.$$

Also,

$$(1.5) \quad \Delta(u_n w) = 0 \quad \text{in } \{u > 0\} \quad \text{and} \quad w_\nu = 0 \quad \text{on } \Gamma, \quad \text{with} \quad w := \frac{u_i}{u_n},$$

where the Neumann condition follows by differentiating the free boundary condition in (1.3) along Γ . Geometrically the quotient w represents the i -derivative of the level set of u viewed as a graph in the e_n direction. Since u_n is harmonic we can

write the equation above as an equation with coefficients in $C^{k,\alpha}$ (see (1.4))

$$\Delta w + 2 \frac{\nabla u_n}{u_n} \cdot \nabla w = 0.$$

Now we apply the Schauder estimates for the Neumann problem and obtain that solutions to (1.5) satisfy $w \in C^{k+2,\alpha}$ and this gives $\Gamma \in C^{k+3,\alpha}$.

It turns out that in the thin one-phase problem the quotient w still satisfies (1.5). We prove Theorem 1.1 by obtaining regularity results as (1.4)-(1.5) in the context of the thin free boundary problem.

To this aim, we consider Schauder estimates at the boundary for harmonic functions in *slit domains*, see Theorem 3.1 for a precise statement. A slit domain is a domain in \mathbb{R}^{n+1} from which we remove an n -dimensional set $\mathcal{P} \subset \{x_{n+1} = 0\}$ (slit), with $C^{k+2,\alpha}$ boundary in \mathbb{R}^n , $\Gamma := \partial_{\mathbb{R}^n} \mathcal{P}$, $k \geq 0$.

In the simplest case when $n = 1$ and \mathcal{P} is the negative x_1 -axis, then a harmonic function u in $B_1 \setminus \mathcal{P}$, even with respect to the x_1 -axis and which vanishes continuously on \mathcal{P} , can be written near the origin as a series of homogenous harmonic functions

$$r^q \cos(q\theta), \quad q = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots,$$

where r and θ denote the polar coordinates. In particular it follows that u has an expansion at the origin of the type

$$u = U_0 (P(x_1, r) + O(r^{k+1+\alpha})), \quad U_0 := r^{\frac{1}{2}} \cos(\theta/2),$$

for some polynomial P of degree $k+1$, where U_0 denotes the first homogenous harmonic function.

In Theorem 3.3 we show that this expansion remains valid also for slit domains in \mathbb{R}^{n+1} with boundary $\Gamma \in C^{k+2,\alpha}$, with P a polynomial of degree $k+1$ in x_1, \dots, x_n and r . In this case (r, θ) denote the polar coordinates with respect to Γ .

In our next step, we use this expansion for u and obtain Schauder estimates at the boundary for solutions w to the Neumann problem (1.5). This equation is quite degenerate in our case since the weight u_n is singular on Γ . We show that w can be approximated in a $C^{k+2,\alpha}$ fashion by a polynomial in the variables x_1, \dots, x_n and r , see Theorem 4.1 for a precise statement.

The proofs of the Schauder estimates above use perturbation arguments, see [CC]. Roughly speaking, we approximate our solutions in dyadic balls by solutions to “constant coefficient” equations which in turn are approximated by appropriate polynomials in x and r .

It is worth remarking that the equations we consider do not behave well under general smooth changes of coordinates. From the expansion of u near Γ we see that one should consider changes of variables which leave r and θ invariant, at least infinitesimally on Γ . For example, if we flatten the boundary Γ and move isometrically the 2D planes perpendicular to Γ , then this change of variables has a loss of one derivative with respect to the smoothness of Γ . In the proof of Theorem 4.1 we use Whitney’s extension theorem to overcome this technical difficulty.

The paper is organized as follows. In Sections 2,3 and 4 we introduce notation and state our main theorems from which Theorem 1.1 follows. Section 5 and 6 are devoted to the proof of our Schauder estimate for solutions to Laplace’s equation in slit domains. Section 7 provides the proof of the Schauder estimate for solutions to the Neumann problem. Some technical facts are proved in the Appendix.

2. NOTATION AND DEFINITIONS

2.1. Notation. We introduce some notation that we use throughout the paper.

Let Γ be a $C^{k+2,\alpha}$ surface in \mathbb{R}^n , $k \geq 0$. Assume for simplicity that Γ is given by the graph of a function g of $n-1$ variables

$$(2.1) \quad \Gamma := \{(x', g(x'))\}, \quad g : B'_1 \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R},$$

satisfying

$$g(0) = 0, \quad \nabla_{x'} g(0) = 0, \quad \|g\|_{C^{k+2,\alpha}(B'_1)} \leq 1.$$

Let \mathcal{P} denote the n dimensional slit in \mathbb{R}^{n+1} given by

$$\mathcal{P} := \{X = (x, x_{n+1}) \in B_1 \mid x_{n+1} = 0, x_n \leq g(x')\}.$$

Notice that in the n dimensional ball $B'_1 \times \{0\}$ we have $\partial_{\mathbb{R}^n} \mathcal{P} = \Gamma$.

Given a point $X = (x, x_{n+1})$ we denote by d the signed distance in \mathbb{R}^n from x to Γ with $d > 0$ above Γ (in the e_n direction). Denote by

$$r := \sqrt{x_{n+1}^2 + d^2}$$

the distance in \mathbb{R}^{n+1} from X to Γ . We have

$$(2.2) \quad \nabla_x r = \frac{d}{r} \nu, \quad \nu = \nabla_x d,$$

and $\nu(x)$ represents the unit normal in \mathbb{R}^n to the parallel surface to Γ passing through x .

Let $\theta \in (-\pi, \pi]$ be the angle between the segment of length r from X to Γ and the x -hyperplane and define

$$U_0(X) := r^{1/2} \cos \frac{\theta}{2} = \frac{1}{\sqrt{2}} \sqrt{d+r}.$$

It is easy to check that

$$(2.3) \quad \nabla_x U_0 = \frac{U_0}{2r} \nu.$$

We denote by c, C various positive constants that depend only on n, k and α .

2.2. The class $C_{xr}^{k,\alpha}$. In this paper we work with functions which near Γ can be expanded as power series in the variables x_1, x_2, \dots, x_n and r . Since we deal with higher regularity we remark that these functions are not sufficiently regular when viewed in the original variable $X = (x, x_{n+1})$. Thus we need to introduce the notion of a $C^{k,\alpha}$ function in the (x, r) -variables. We are interested only in power expansions at points on Γ and for this reason we define the notion of pointwise $C^{k,\alpha}$ function in the (x, r) variables.

We denote by

$$P(x, r) = a_{\mu m} x^\mu r^m, \quad \deg P = k,$$

a polynomial of degree k in the (x, r) variables, and we use throughout the paper the summation convention over repeatedly indices. Above we used the following notation:

$$x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n}, \quad |\mu| = \mu_1 + \dots + \mu_n, \quad \mu_i \geq 0.$$

Sometimes it is useful to think that $a_{\mu m}$ are defined for all indices (μ, m) , by extending them to be 0.

We also denote

$$\|P\| := \max |a_{\mu m}|.$$

Definition 2.1. We say that a function $f : B_1 \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is *pointwise* $C^{k,\alpha}$ in the (x, r) -variables at $0 \in \Gamma$ and write $f \in C_{xr}^{k,\alpha}(0)$ if there exists a (tangent) polynomial $P_0(x, r)$ of degree k such that

$$f(X) = P_0(x, r) + O(|X|^{k+\alpha}).$$

We define $\|f\|_{C_{xr}^{k,\alpha}(0)}$ as the smallest constant M such that

$$\|P_0\| \leq M, \quad \text{and} \quad |f(X) - P_0(x, r)| \leq M|X|^{k+\alpha},$$

for all X in the domain of definition.

Similarly, we may write the definition for f to be pointwise $C_{xr}^{k,\alpha}$ at some other point $Z \in \Gamma$. Next we define the notion of $C_{xr}^{k,\alpha}$ on a whole subset $K \subset \Gamma$.

Definition 2.2. Let $K \subset \Gamma$. We say that $f \in C_{xr}^{k,\alpha}(K)$ if there exists a constant M such that $f \in C_{xr}^{k,\alpha}(Z)$ for all $Z \in K$ and $\|f\|_{C_{xr}^{k,\alpha}(Z)} \leq M$ for all $Z \in K$.

The smallest M in the definition above is denoted by $\|f\|_{C_{x,r}^{k,\alpha}(K)}$.

3. HARMONIC FUNCTIONS IN SLIT DOMAINS

As first step towards the proof of our main Theorem 1.1, we are interested in the regularity of solutions to the Laplace equation in slit domains and their precise behavior on the edge of the slit. We collect here our main statements. First we remark that we may restrict ourself to the case when solutions are even with respect to x_{n+1} . Indeed, let

$$\Delta u = F \quad \text{in} \quad B_1 \setminus \mathcal{P},$$

and u vanish continuously on \mathcal{P} . We decompose $u = u_{ev} + u_{od}$ with u_{ev} , u_{od} even respectively odd with respect to x_{n+1} . Notice that u_{ev} and u_{od} solve the Laplace equation with right hand side F_{ev} , respectively F_{od} . Since u_{od} vanishes continuously on $x_{n+1} = 0$, its regularity follows from the boundary regularity for Laplace equation in smooth domains. For example if F_{od} is smooth, then u_{od} can be expanded as a power series in x , x_{n+1} at the origin.

Next we state our Schauder estimates in slit domains with $C^{k+2,\alpha}$ boundary.

Let $u \in C(B_1)$ be even in the x_{n+1} coordinate, with $\|u\|_{L^\infty} \leq 1$, and

$$(3.1) \quad \begin{cases} \Delta u = \frac{U_0}{r} f & \text{in } B_1 \setminus \mathcal{P} \\ u = 0 & \text{on } \mathcal{P}. \end{cases}$$

Theorem 3.1 (Schauder estimates in slit domains). *Let Γ , u satisfy (2.1), (3.1) with*

$$f \in C_{xr}^{k,\alpha}(\Gamma \cap B_1), \quad \|f\|_{C_{xr}^{k,\alpha}(\Gamma \cap B_1)} \leq 1.$$

Then,

$$(3.2) \quad \left\| \frac{u}{U_0} \right\|_{C_{xr}^{k+1,\alpha}(\Gamma \cap B_{1/2})} \leq C$$

and

$$(3.3) \quad \left\| \frac{\nabla_x u}{(U_0/r)} \right\|_{C_{xr}^{k+1,\alpha}(\Gamma \cap B_{1/2})} \leq C$$

with C a constant depending only on n , k and α .

The Theorem above states that u satisfies the following expansion at $0 \in \Gamma$

$$u(X) = U_0(X) (P_0(x, r) + O(|X|^{k+1+\alpha})),$$

for some polynomial $P_0(x, r)$ of degree $k+1$. The derivatives u_i are in fact obtained by differentiating formally this expansion in the x_i direction. Using (2.2)-(2.3) we have

$$(3.4) \quad \nabla_x u = \frac{U_0}{r} \left[\frac{1}{2} P_0 \nu + r \partial_x P_0 + (\partial_r P_0) d \nu + O(|X|^{k+1+\alpha}) \right].$$

Since $\nu, d \in C_x^{k+1, \alpha}$ we obtain

$$u_i = \frac{U_0}{r} (P_0^i(x, r) + O(|X|^{k+1+\alpha})), \quad \deg P_0^i = k+1,$$

for some polynomial P_0^i .

The boundary Harnack estimate for harmonic functions in slit domains with Lipschitz boundary (in \mathbb{R}^n) states that the quotient of two positive solutions which vanish continuously on \mathcal{P} is Hölder continuous (see [CFMS]). Theorem 3.1 can be understood as an optimal boundary Harnack estimate in the case when the boundary of the slit Γ has higher $C^{k, \alpha}$ regularity.

We prove Theorem 3.1 in Section 5 for the case $k = 0$ and in Section 6 for general k . We mention that the theorem holds also for $k = -1$, i.e. when $\Gamma \in C^{1, \alpha}$, see [DS3].

Remark 3.2. In Section 5, we will show also that if f is more regular away from Γ , say such that it guarantees the existence of second derivatives of u locally, then also u_{ij} are obtained by formally differentiating the expression above. In particular if f is a C^α function in the X variable in the whole B_1 , then

$$u_{ij} = \frac{U_0}{r^3} (P_0^{ij}(x, r) + O(|X|^{k+2+\alpha})),$$

with P_0^{ij} a sum of monomials with degrees between 1 and $k+2$.

In the case of harmonic functions ($f \equiv 0$) we can obtain all derivatives of order $|\mu| \leq k+2$ by differentiating formally,

$$D_x^\mu u = \frac{U_0}{r^{2|\mu|-1}} (P_0^\mu(x, r) + O(|X|^{k+|\mu|+\alpha})),$$

with P_0^μ having monomials with degrees between $|\mu| - 1$ and $k + |\mu|$.

We also state the polynomial expansion near Γ for general harmonic functions (not necessarily even) in slit domains with $C^{k+2, \alpha}$ boundary, since it is of interest on its own. The expansion involves the first two harmonic functions in 2D i.e.

$$U_0 = r^{\frac{1}{2}} \cos(\theta/2) \quad \text{even}, \quad x_{n+1} = r \sin \theta \quad \text{odd}$$

multiplied by powers of r . Precisely we have

Theorem 3.3 (Expansion of harmonic functions). *Assume $\Gamma \in C^{k+2, \alpha}$, $k \geq 0$, and $u \in C(B_1)$ satisfies*

$$\Delta u = 0 \quad \text{in } B_1 \setminus \mathcal{P}, \quad u = 0 \quad \text{on } \mathcal{P}.$$

There exist functions

$$a_j(x) \in C^{k+1-j, \alpha} \quad \text{and} \quad b_j(x) \in C^\infty,$$

such that for all $X \in B_{1/2}$

$$\left| u(X) - U_0 \left(\sum_{j=0}^{k+1} a_j(x) r^j \right) - x_{n+1} \left(\sum_{j \leq k/2} b_j(x) x_{n+1}^{2j} \right) \right| \leq M U_0 r^{k+1+\alpha}.$$

The constant M and the norms of a_j , b_j depend on $\|u\|_{L^\infty}$, $\|\Gamma\|_{C^{k+2,\alpha}}$, k , α and n .

The first and second term above approximate the even respectively odd part of u . More generally we will show that if

$$\Delta u = F \quad \text{in } B_1 \setminus \mathcal{P}, \quad F \in C^{k,\alpha},$$

then

$$u = U_0 \left(\sum_{m=0}^{k+1} a_m(x) r^m + O(r^{k+1+\alpha}) \right) + x_{n+1} \left(\sum_{m=0}^k b_m(x) x_{n+1}^m \right),$$

for functions $a_m \in C^{k+1-m,\alpha}$, $b_m \in C^{k-m,\alpha+\frac{1}{2}}$.

4. THE THIN ONE-PHASE PROBLEM

In this Section we show that our main Theorem 1.1 follows from Theorem 3.1 and a Schauder estimate for a Neumann-type problem which we also state here.

Assume $u \in C(B_1)$ is a solution to the thin one-phase free boundary problem

$$(4.1) \quad \begin{cases} \Delta u = 0 & \text{in } B_1 \setminus \mathcal{P}, \\ \frac{\partial u}{\partial U_0} = 1 & \text{on } \Gamma, \end{cases} \quad u = 0 \quad \text{on } \mathcal{P},$$

where

$$\frac{\partial u}{\partial U_0}(Z) := \lim_{t \rightarrow 0^+} \frac{u(z + t\nu, 0)}{t^{1/2}}, \quad Z = (z, 0) \in \Gamma.$$

We assume that $\Gamma \in C^{2,\alpha}$ satisfies (2.1), and after replacing u by its even part, we also assume that u is even in x_{n+1} . By Theorem 3.1, at a point $Z \in \Gamma$ we have the expansion

$$u(X) = U_0(X) (P_Z(x, r) + O(|X - Z|^{1+\alpha})), \quad \deg P = 1.$$

Notice that

$$P_Z(z, 0) = \frac{\partial u}{\partial U_0}(Z).$$

4.1. Equation for the quotient w . We show that the quotient

$$w := \frac{u_i}{u_n}, \quad w \in C(B_1)$$

satisfies the following problem with Neumann boundary condition on Γ :

$$(4.2) \quad \begin{cases} \Delta(u_n w) = 0 & \text{in } B_1 \setminus \mathcal{P}, \\ w_\nu = 0 & \text{on } \Gamma, \end{cases}$$

with

$$w_\nu(Z) := \lim_{t \rightarrow 0^+} \frac{w(z + t\nu, 0) - w(z, 0)}{t}, \quad Z \in \Gamma.$$

Notice that w represents the derivative in the $-e_i$ direction of the level sets of u viewed as graphs in the e_n direction. In particular on Γ

$$(4.3) \quad w(Z) = -g_i(z'),$$

and this gives the relation between the regularity of w on Γ and the regularity of Γ itself.

First, we remark that w is indeed continuous in B_1 . In fact from (3.4) it follows that $u_n > 0$ is a neighborhood of zero. After a dilation we can assume that this is true in B_1 . Now, again from (3.4) we conclude that w is continuous on Γ , and boundary Harnack inequality gives the continuity of w on the slit \mathcal{P} .

Next we check the Neumann condition for w . Let

$$P_0(x, r) = a_0 + a_i x_i + a_{n+1} r$$

be the polynomial in the expansion of u at 0. From the free boundary condition we find,

$$1 = P_0(z, 0) + O(|z|^{1+\alpha}), \quad Z \in \Gamma$$

thus, using that $\nabla_{x'} g(0) = 0$, we get

$$a_0 = 1, \quad a_i = 0 \quad 1 \leq i \leq n-1.$$

By (3.4) we see that on the line te_n , $\nu = e_n$ hence

$$u_i(te_n) = t^{-\frac{1}{2}} O(t^{1+\alpha}), \quad 1 \leq i \leq n-1,$$

which gives

$$(t^{\frac{1}{2}} u_i)(0) = \frac{d}{dt} (t^{\frac{1}{2}} u_i)(0) = 0.$$

For any vector $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$, $\tau_n \neq 0$ we obtain that

$$\frac{d}{dt} \left[\log(t^{\frac{1}{2}} u_\tau) \right] (0) \quad \text{does not depend on } \tau,$$

and it follows that for any two vectors in \mathbb{R}^n , τ, σ , with $\tau_n \neq 0$ we have

$$\left(\frac{u_\sigma}{u_\tau} \right)_{e_n} (0) = 0.$$

Thus w solves the Neumann problem (4.2). We will prove the following estimate for solutions to such Neumann problem.

Theorem 4.1. *Let $\Gamma \in C^{k+2, \alpha}$ satisfy (2.1) and let u be a harmonic function in $B_1 \setminus \mathcal{P}$, even in x_{n+1} , such that $\frac{1}{2} U_0 \leq u \leq 2U_0$. Assume $w \in C(B_1)$, even in x_{n+1} , solves the Neumann problem*

$$\begin{cases} \Delta(u_n w) = 0 & \text{in } B_1 \setminus \mathcal{P}, \\ w_\nu = 0 & \text{on } \Gamma. \end{cases}$$

Then $w \in C_{x,r}^{k+2, \alpha}(\Gamma)$ and

$$\|w\|_{C_{x,r}^{k+2, \alpha}(\Gamma \cap B_{1/2})} \leq C \|w\|_{L^\infty(B_1)},$$

with C depending only on n, k, α .

Clearly the function w is a $C^{k+2, \alpha}$ function when restricted to Γ .

Theorem 1.1 is a direct corollary of Theorem 3.1 and Theorem 4.1. Indeed, if u is a solution to the thin one-phase problem and $\Gamma \in C^{k+2, \alpha}$, then by Theorem 3.1 and the free boundary condition, u satisfies the assumptions of Theorem 4.1 (after a dilation.) We then apply Theorem 4.1 to the quotient $w = u_i/u_n$ and obtain (see (4.3)) that in fact $\Gamma \in C^{k+3, \alpha}$.

Remark 4.2. From the proof of Theorem 4.1 it follows that the conclusion holds if the homogenous Neumann condition is replaced by $w_\nu \in C^{k+1,\alpha}$ on Γ .

Remark 4.3. Theorems 3.1 and 4.1 apply also in the case $k = -1$, that is when $\Gamma \in C^{1,\alpha}$ (see [DS3]). However the Neumann condition for w cannot be justified in this case. This is the main reason why we require initially $\Gamma \in C^{2,\alpha}$.

4.2. General setting. Assume $u \in C(B_1)$ satisfies the thin one-phase problem with general free boundary condition

$$(4.4) \quad \begin{cases} \Delta u = 0 & \text{in } B_1 \setminus \mathcal{P}, & u = 0 & \text{on } \mathcal{P}, \\ \frac{\partial u}{\partial U_0} = G(z) & \text{on } \Gamma, \end{cases}$$

with $G > 0$, $G \in C^{k+2,\alpha}$.

If $\Gamma \in C^{k+2,\alpha}$ then the quotient $w = u_i/u_n$ satisfies a Neumann condition

$$w_\nu = h \quad \text{on } \Gamma,$$

for some $h \in C^{k+1,\alpha}$ depending on ν , G and the derivatives of G .

Indeed, as above, at the origin we find ($1 \leq i \leq n-1$)

$$a_0 = G, \quad a_i = G_{z_i},$$

where G and its derivatives are evaluated at 0. Then on the line te_n we obtain

$$(t^{\frac{1}{2}}u_i)(0) = 0, \quad (t^{\frac{1}{2}}u_i)_{e_n}(0) = G_{z_i} \quad (t^{\frac{1}{2}}u_n)(0) = G(0)/2,$$

and now it is straightforward to obtain the dependence of h on ν , G , ∇G . Using Remark 4.2 we obtain optimal regularity of the free boundary in problem (4.4).

Proposition 4.4. *Assume u satisfies (4.4) for a positive $G \in C_x^{k+2,\alpha}$, for some $k \geq 0$ and $\alpha \in (0, 1)$. If $\Gamma \in C^{2,\alpha}$ then $\Gamma \in C^{k+3,\alpha}$.*

4.3. Constant coefficients. We prove our theorems using the estimates for the “constant coefficients” case together with perturbation arguments. Precisely, Theorems 3.1 and 4.1 rely on the following two theorems.

Theorem 4.5. *Assume $\Gamma = \{x_n = 0\}$ and $u \in C(B_1)$ is even, $\|u\| \leq 1$ and satisfies*

$$\Delta u = 0 \quad \text{in } B_1 \setminus \mathcal{P}, \quad u = 0 \quad \text{on } \mathcal{P}.$$

For any $k \geq 0$, there exists a polynomial $P_0(x, r)$ of degree k such that $U_0 P_0$ is harmonic in $B_1 \setminus \mathcal{P}$ and

$$|u - U_0 P_0| \leq C|X|^{k+1}U_0,$$

for some constant C depending on k and n .

Theorem 4.6. *Assume $\Gamma = \{x_n = 0\}$ and $w \in C(B_1)$, $\|w\| \leq 1$ satisfies*

$$(4.5) \quad \Delta((U_0)_n w) = 0, \quad \text{in } B_1 \setminus \mathcal{P}, \quad w_\nu = 0 \quad \text{on } \Gamma.$$

For any $k \geq 0$, there exists a polynomial $T(x, r)$ of degree k , of the form

$$T = Q(x') + r P(x, r), \quad \deg P = k - 1,$$

such that T satisfies (4.5) and

$$|w - T| \leq C|X|^{k+1},$$

for some constant C depending on k and n .

The proofs of these two theorems are postponed till the appendix. They use the linearity and the translation invariance in the x' direction of the corresponding equations.

5. POINTWISE SCHAUDER ESTIMATE

In this section we present our key estimate, that is a pointwise Schauder estimate in slit domains. We prove it under rather general assumptions. Theorems 3.1 and 3.3 will easily follow from this result.

Proposition 5.1 (Pointwise Schauder estimate). *Assume that $u \in C(B_1)$ is even and vanishes on \mathcal{P} , $\|u\|_{L^\infty} \leq 1$, and*

$$(5.1) \quad \Delta u(X) = \frac{U_0}{r} R(x, r) + F(X) \quad \text{in } B_1 \setminus \mathcal{P},$$

with

$$|F(X)| \leq r^{-\frac{1}{2}} |X|^{k+\alpha} \quad \text{and } R(x, r) \text{ a polynomial of degree } k \text{ with } \|R\| \leq 1.$$

There exists a polynomial $P_0(x, r)$ of degree $k+1$ with coefficients bounded by C such that

$$\left| \frac{u}{U_0} - P_0 \right| \leq C |X|^{k+1+\alpha},$$

and

$$|\Delta(u - U_0 P_0)| \leq C r^{-\frac{1}{2}} |X|^{k+\alpha} \quad \text{in } B_1 \setminus \mathcal{P},$$

with C depending on k, α, n .

The proof of Proposition 5.1 is similar to the proof of the classical pointwise Schauder estimates, but in our case we work with monomials $U_0 x^\mu r^\gamma$ instead of monomials of the type $x^\mu x_{n+1}^\gamma$. The reason is that monomials $U_0 x^\mu r^\gamma$ remain of the “same form” after applying Δ .

Indeed, first notice that in a $2D$ plane (t, x_{n+1}) with $r = \sqrt{t^2 + x_{n+1}^2}$ we have

$$(5.2) \quad \begin{aligned} \Delta_{t, x_{n+1}}(r^m U_0) &= m(m+1) r^{m-2} U_0, \\ \partial_t(r^m U_0) &= U_0 \left(\frac{1}{2} r^{m-1} + m t r^{m-2} \right). \end{aligned}$$

Therefore in \mathbb{R}^{n+1} we obtain

$$(5.3) \quad \Delta(r^m U_0) = m(m+1) r^{m-2} U_0 + \kappa(x) \partial_t(r^m U_0)$$

with $\kappa(x)$ the mean curvature of the parallel surface to Γ passing through x . We also denote by $\nu(x)$ the normal to this parallel surface. Thus,

$$\kappa(x) = -\Delta d \in C_x^{k, \alpha}, \quad \nu(x) = \nabla d \in C_x^{k+1, \alpha}.$$

To fix ideas, we present the proof of Proposition 5.1 first in the case $k = 0$. Then we explain the general case.

5.1. Proof of Proposition 5.1 in the case $k = 0$. We remark that in this case R is a constant. After performing an initial dilation, we may assume that our hypotheses in B_1 are

$$\|\Gamma\|_{C^{2,\alpha}} \leq \delta, \quad |R| \leq \delta, \quad |F| \leq \delta r^{-\frac{1}{2}} |X|^\alpha,$$

for some δ small, to be made precise later.

From the formulas above, we have

$$\begin{aligned} \Delta U_0 &= \frac{1}{2} \kappa(x) \frac{U_0}{r}, \\ \Delta(rU_0) &= \left(2 + \left(d + \frac{1}{2}r\right) \kappa(x)\right) \frac{U_0}{r}, \end{aligned}$$

and we easily compute

$$\Delta(x_i U_0) = \nu^i \frac{U_0}{r}.$$

If

$$P(X) = a_0 + \sum_{i=1}^n a_i x_i + a_{n+1} r,$$

then

$$(5.4) \quad \Delta(U_0 P) = \frac{U_0}{r} \left(\frac{\kappa(0)}{2} a_0 + a_n + 2a_{n+1} + h_0(x) + r h_1(x) \right)$$

with

$$h_0, h_1 \in C_x^\alpha, \quad h_0(0) = 0, \quad \|h_0\|_{C^\alpha}, \|h_1\|_{C^\alpha} \leq C\delta \|P\|.$$

We say that P is an *approximating polynomial* for equation (5.1) at 0, if

$$\frac{\kappa(0)}{2} a_0 + a_n + 2a_{n+1} = R.$$

We prove Proposition 5.1 by approximating u in a sequence of balls B_{ρ^m} with appropriate functions $U_0 P_m$ with P_m approximating polynomials.

It suffices to prove the next lemma.

Lemma 5.2. *There exist universal constants ρ, δ depending only on α and n , such that if P with $\|P\| \leq 1$ is an approximating polynomial for u in B_λ , that is P is approximating for (5.1) at 0 and*

$$|u - U_0 P|_{L^\infty(B_\lambda)} \leq \lambda^{3/2+\alpha},$$

for some $\lambda > 0$, then there exists an approximating polynomial \bar{P} for u in $B_{\rho\lambda}$:

$$|u - U_0 \bar{P}|_{L^\infty(B_{\rho\lambda})} \leq (\rho\lambda)^{3/2+\alpha}, \quad \|\bar{P} - P\|_{L^\infty(B_\lambda)} \leq C\lambda^{1+\alpha}.$$

Proof. Define \tilde{u} to be the error between u and $U_0 P$ rescaled at unit size, that is

$$u - U_0 P =: \lambda^{\frac{3}{2}+\alpha} \tilde{u} \left(\frac{X}{\lambda} \right).$$

Then our assumption reads $\|\tilde{u}\|_{L^\infty(B_1)} \leq 1$. Since u solves (5.1),

$$F + \frac{U_0}{r} R - \Delta(U_0 P) = \lambda^{-\frac{1}{2}+\alpha} \Delta \tilde{u} \left(\frac{X}{\lambda} \right),$$

thus using (5.4) and that P is an approximating polynomial we obtain

$$(5.5) \quad \Delta \tilde{u}\left(\frac{X}{\lambda}\right) = \lambda^{\frac{1}{2}-\alpha} \left(F(X) - \frac{U_0}{r}(h_0(x) + rh_1(x)) \right).$$

Using the hypothesis on F we find

$$|\Delta \tilde{u}(X)| \leq C\delta r^{-\frac{1}{2}} \quad \text{in } B_1.$$

Denote by $\tilde{\Gamma}$, $\tilde{\mathcal{P}}$, \tilde{U}_0 the rescalings of Γ , \mathcal{P} and U_0 from B_λ to B_1 i.e.

$$\tilde{\Gamma} := \frac{1}{\lambda}\Gamma, \quad \tilde{\mathcal{P}} := \frac{1}{\lambda}\mathcal{P}, \quad \tilde{U}_0(X) := \lambda^{-\frac{1}{2}}U_0(\lambda X).$$

We decompose \tilde{u} as

$$\tilde{u} = \tilde{u}_0 + \tilde{v}$$

with

$$\begin{cases} \Delta \tilde{u}_0 = 0 & \text{in } B_1 \setminus \tilde{\mathcal{P}}, \\ \tilde{u}_0 = \tilde{u} & \text{on } \partial B_1 \cup \tilde{\mathcal{P}}, \end{cases}$$

and

$$\begin{cases} |\Delta \tilde{v}| \leq C\delta r^{-\frac{1}{2}} & \text{in } B_1 \setminus \tilde{\mathcal{P}}, \\ \tilde{v} = 0 & \text{on } \partial B_1 \cup \tilde{\mathcal{P}}. \end{cases}$$

Using barriers we can show the following

$$(5.6) \quad \|\tilde{v}\|_{L^\infty(B_1)} \leq C\delta \tilde{U}_0.$$

We postpone the proof of (5.6) till later.

To estimate \tilde{u}_0 we observe that \tilde{u}_0 is a harmonic function in $B_1 \setminus \tilde{\mathcal{P}}$, $|\tilde{u}_0| \leq 1$ and as $\delta \rightarrow 0$, $\tilde{\Gamma}$ converges in the $C^{2,\alpha}$ norm to the hyperplane $\{x_n = 0\}$. Moreover, \tilde{u}_0 is uniformly Hölder continuous in $B_{1/2}$. By compactness, if δ is sufficiently small universal, \tilde{u}_0 can be approximated in $B_{1/2}$ by a solution of the Laplace problem with $\Gamma = \{x_n = 0\}$. Thus by Theorem 4.5,

$$(5.7) \quad \|\tilde{u}_0 - \tilde{U}_0 Q\|_{L^\infty(B_\rho)} \leq C\rho^{2+\frac{1}{2}}$$

with $\|Q\| \leq C$, and since $U_0 Q$ is harmonic we also get from (5.4) that

$$Q = b_0 + b_i x_i + b_{n+1} r, \quad 2b_{n+1} + b_n = 0.$$

Using also (5.6) we find

$$\|\tilde{u} - \tilde{U}_0 Q\|_{L^\infty(B_\rho)} \leq C\rho^{\frac{3}{2}} + C\delta \leq \frac{1}{2}\rho^{\frac{3}{2}+\alpha}$$

provided that we choose first ρ and then δ , universal, sufficiently small.

Writing this inequality in terms of the original function u we find,

$$|u - U_0(P + \lambda^{1+\alpha}Q(\frac{X}{\lambda}))| \leq \frac{1}{2}(\lambda\rho)^{\frac{3}{2}+\alpha} \quad \text{in } B_{\rho\lambda}.$$

However $P(X) + \lambda^{1+\alpha}Q(X/\lambda)$ is not an approximating polynomial and therefore we need to perturb Q by a small amount. Let

$$\bar{Q} := Q - \frac{1}{4}\kappa(0)b_0\lambda r$$

thus $P + \lambda^{1+\alpha}\bar{Q}(X/\lambda)$ is approximating. Notice that

$$\|Q - \bar{Q}\| \leq C\delta$$

and therefore we can replace Q by \bar{Q} in (5.7) and obtain the same conclusion.

We define

$$\bar{P} = P + \lambda^{1+\alpha} \bar{Q}\left(\frac{x}{\lambda}\right),$$

thus

$$\|\bar{P} - P\|_{L^\infty(B_\lambda)} \leq C\lambda^{1+\alpha}.$$

This concludes the proof of the lemma. \square

We can now conclude the proof of Proposition 5.1.

After multiplying u by a small constant, we see that the hypotheses of the lemma are satisfied for some initial λ_0 small with $P = Rx_n$. Now we may iterate the lemma for all $\lambda = \lambda_0 \rho^m$ and conclude that there exists a limiting approximating polynomial P_0 , $\|P_0\| \leq C$, such that

$$|u - U_0 P_0| \leq C|X|^{\frac{3}{2}+\alpha} \quad \text{in } B_1.$$

In B_λ we may argue as in the proof above with P_0 replacing P and obtain

$$|\tilde{u}| \leq |\tilde{u}_0| + |\tilde{v}| \leq C\tilde{U}_0 \quad \text{in } B_{1/2},$$

where we have used boundary Harnack inequality for \tilde{u}_0 and (5.6) for \tilde{v} . Thus,

$$\|u - U_0 P_0\|_{L^\infty(B_\lambda)} \leq C\lambda^{1+\alpha} U_0.$$

Moreover, since P_0 is approximating, by (5.4)

$$\Delta(u - U_0 P_0) = F(X) + \frac{U_0}{r}(h_0(x) + rh_1(x)) = O(r^{-\frac{1}{2}}|X|^\alpha).$$

We are left with the proof of (5.6).

Proof of claim (5.6). We use as lower (upper) barriers multiples of the function

$$\bar{v} := -U_0 + U_0^2.$$

Notice that $\bar{v} \leq 0$ in B_1 . In the 2D plane (t, x_{n+1}) we have

$$\Delta \bar{v} \geq 2|\nabla U_0|^2 \geq cr^{-1}, \quad |\partial_t \bar{v}| \leq Cr^{-\frac{1}{2}},$$

thus in \mathbb{R}^{n+1} we also satisfy

$$\Delta \bar{v} \geq cr^{-1}.$$

\square

We present some remarks which we often use about functions $w \in C_{xr}^{k,\alpha}(\Gamma)$. Assume for simplicity that $k = 1$ since the general case follows similarly.

Remark 5.3. Let P_0 and P_Z be the tangent polynomials for w at 0 and $Z \in \Gamma$ with $|Z| = \lambda$. Since both P_0 and P_Z approximate w in $B_{\lambda/2}(\lambda e_n)$ with a $C\lambda^{1+\alpha}$ error, then

$$\|P_0 - P_Z\|_{L^\infty(B_{2\lambda})} \leq C\lambda^{1+\alpha}$$

and this implies that the free coefficients of P_0 and P_Z differ by $C|Z|^{1+\alpha}$ and the first order coefficients differ by $C|Z|^\alpha$.

For general k we obtain that the corresponding coefficients of the monomials of degree m for P_0 and P_Z differ by $C|Z|^{k-m+\alpha}$.

Notice that we only used that P_0 (respectively P_Z) approximates w in a cone around the corresponding normal to Γ , say $\{|X| \leq x_n\}$.

Remark 5.4. Let W be the function $W(X) = P_Z(x, r)$ where Z denotes the projection of x onto Γ . In other words W coincides with the tangent polynomial on each 2D plane perpendicular to Γ . Then

$$W(X) = a_0(Z) + a_n(Z)d + a_{n+1}(Z)r, \quad w = W + O(r^{1+\alpha}),$$

for some functions a_0, a_n, a_{n+1} defined on Γ . Thus, w and W have the same tangent polynomials on Γ . Now it is not difficult to show that $a_0 \in C^{1,\alpha}$, $a_n \in C^\alpha$, $a_{n+1} \in C^\alpha$.

For general k we find that W is a polynomial of degree k in (d, r) with coefficients depending on Z . The monomials of degree m in (d, r) have coefficients in $C^{k-m,\alpha}(\Gamma)$.

5.2. Applications of Proposition 5.1 and Proof of Theorem 3.1. It is clear that the statement (3.2) in Theorem 3.1 follows from the pointwise estimate in Proposition 5.1 applied with $f(X) = R(x, r) + h(X)$ with R a polynomial of degree k and $h(X) = O(|X|^{k+\alpha})$. To obtain (3.3), we need to deduce some consequences of Proposition 5.1 in which we estimate the derivatives of u near Γ . Roughly speaking we can estimate ∇u by differentiating formally the expansion of u . However in order to do this we need to impose slightly more regularity on the right hand side F . First we notice that, by scaling, we can estimate the derivatives of u from the conclusion of Proposition 5.1 in non-tangential cones to Γ .

Lemma 5.5. *Assume that u satisfies the hypotheses of Proposition 5.1. Then*

$$(5.8) \quad \left| u_i - \frac{U_0}{r} P_0^i \right| \leq C |X|^{\frac{1}{2} + \alpha + k} \quad \text{in the cone } \{r \geq |x'|\},$$

with $\deg P_0^i = k + 1$, and $(U_0/r)P_0^i$ is obtained by formally differentiating $U_0 P_0$ at the origin in the x_i direction.

Remark 5.6. If the hypotheses of Proposition 5.1 are satisfied at all points $Z \in \Gamma \cap B_{1/2}$ instead of only the origin then we obtain that the inequality (5.8) holds in fact for all X in a neighborhood of the origin. This follows easily by applying the arguments of Remark 5.3 to u_i .

Proof. We assume $k = 0$. As in the proof of Proposition 5.1 denote by \tilde{u} the rescaling of $u - U_0 P_0$ from B_λ to B_1 i.e.

$$u - U_0 P_0 = \lambda^{\frac{3}{2} + \alpha} \tilde{u}(X/\lambda),$$

thus

$$\Delta \tilde{u} = \tilde{F}, \quad \|\tilde{u}\|_{L^\infty(B_1)} \leq C,$$

with

$$(5.9) \quad \tilde{F}(X) := \lambda^{\frac{1}{2} - \alpha} F(\lambda X) + \frac{\tilde{U}_0}{r} (\lambda^{-\alpha} h_0(\lambda x) + \lambda^{1-\alpha} r h_1(\lambda x)).$$

Let \mathcal{C} denote the conical domain

$$\mathcal{C} := \{r \geq 2|x'|\} \cap (B_1 \setminus B_{1/4}).$$

Then

$$\|\tilde{F}\|_{L^\infty(\mathcal{C})} \leq C,$$

hence

$$(5.10) \quad |\nabla_x \tilde{u}| \leq C \quad \text{in } \mathcal{C}' := \{r \geq |x'|\} \cap (B_{3/4} \setminus B_{1/2}).$$

This gives, for all $\lambda > 0$

$$|\nabla_x(u - U_0 P_0)| \leq C\lambda^{\frac{1}{2}+\alpha} \quad \text{in } \mathcal{C}' := \{r \geq |x'|\} \cap (B_{\frac{3}{4}\lambda} \setminus B_{\frac{1}{2}\lambda}).$$

On the other hand,

$$\nabla_x(U_0 P_0) = \frac{U_0}{r} \left[\frac{1}{2} P_0 \nu + r \nabla_x P_0 + (\partial_r P_0) d\nu \right].$$

Since $\nu, d \in C_x^{1,\alpha}$ we obtain in $\{r \geq |x'|\}$

$$|\partial_i(U_0 P_0) - \frac{U_0}{r} [P_0^i(x, r)]| \leq C \frac{U_0}{r} |X|^{1+\alpha}$$

with $\deg P_0^i = 1$, and this proves Lemma 5.5. \square

We present some variations of Lemma 5.5, which will lead to the proof of the second part of Theorem 3.1 as well.

1) If $\lambda^{\frac{1}{2}-\alpha} F(\lambda X)$ is uniformly Hölder continuous at all points in the conical n -dimensional set $\mathcal{P} \cap \mathcal{C}$ then, since h_0, h_1 are Hölder continuous, \tilde{F} is also uniformly Hölder continuous at all points in this set (see formula (5.9)). Then, since the u_i 's ($1 \leq i \leq n$) vanish on the plate \mathcal{P} , we can improve (5.10) to

$$|\nabla_x u| \leq C \tilde{U}_0 \quad \text{in } \mathcal{C}'.$$

This means that the right hand side in (5.8) can be replaced by $C|X|^\alpha U_0$, that is

$$(5.11) \quad \left| u_i - \frac{U_0}{r} P_0^i \right| \leq C \frac{U_0}{r} |X|^{\alpha+1+k} \quad \text{in the cone } \{r \geq |x'|\}.$$

It is easy to check that this is the case when F has the form $(U_0/r)h$ with $h(0) = 0$ and h pointwise C_X^α at 0.

Now (3.3) in Theorem 3.1 readily follows from (5.11), by decomposing $f(X) = R(x, r) + h(X)$ with R a polynomial of degree k and $h(X) = O(|X|^{k+\alpha})$ and arguing as in Remark 5.6.

2) If $\lambda^{\frac{1}{2}-\alpha} F(\lambda X)$ is uniformly Hölder continuous at all points in \mathcal{C} then we can estimate the second derivatives. Indeed (see (5.9)), $\|\tilde{F}\|_{C^\alpha(\mathcal{C})} \leq C$, thus

$$(5.12) \quad |D_x^2 \tilde{u}| \leq C \quad \text{in } \mathcal{C}'.$$

Since

$$\partial_{ij}(U_0 P_0) = \frac{U_0}{r^3} (P_0^{ij} + O(|X|^{k+2+\alpha})), \quad 1 \leq \deg P_0^{ij} \leq k+2,$$

we obtain

$$\left| u_{ij} - \frac{U_0}{r^3} P_0^{ij} \right| \leq C |X|^{k+\alpha-\frac{1}{2}} \quad \text{in the cone } \{|x'| \leq r\}.$$

In the case $F \equiv 0$ then we can improve this estimate. Indeed, \tilde{F} is now pointwise $C^{1,\alpha}$ and (5.12) can be replaced by

$$(5.13) \quad |D_x^2 \tilde{u}| \leq C \tilde{U}_0 \quad \text{in } \mathcal{C}'.$$

Then, arguing as in part 1) we obtain

$$\left| u_{ij} - \frac{U_0}{r^3} P_0^{ij} \right| \leq C \frac{U_0}{r^3} |X|^{k+1+\alpha}.$$

6. THE PROOF OF PROPOSITION 5.1. THE GENERAL CASE.

The proof is essentially the same as in the case $k = 0$. The difference occurs in the notion of approximate polynomial, since we need to satisfy several linear equations rather than just a single one.

We now proceed to give the definition of approximating polynomial, for this general case.

Let \bar{i} denote the vector of indices μ with 1 on the i th position and zeros elsewhere. Using (5.2), (5.3), we obtain

$$\begin{aligned}\Delta(x^\mu r^m U_0) &= r^m U_0 \Delta(x^\mu) + x^\mu \Delta(r^m U_0) + 2 \nabla x^\mu \cdot \nabla(r^m U_0) \\ &= U_0(r^m \mu_i(\mu_i - 1) x^{\mu - 2\bar{i}} + m(m+1) x^\mu r^{m-2} + \\ &\quad + x^\mu (\frac{1}{2} r^{m-1} + m dr^{m-2}) \kappa(x) + 2(\frac{1}{2} r^{m-1} + m dr^{m-2}) \nu \cdot \nabla_x x^\mu).\end{aligned}$$

By Taylor expansion at 0, we write each ν^i , d and κ as a sum between a polynomial of degree k and a $C^{k,\alpha}$ function in x with vanishing derivatives up to order k at 0. We use that the lowest degree terms in each expansion are

$$(6.1) \quad \nu^i = \delta_n^i + \dots, \quad \kappa = \kappa(0) + \dots, \quad d = x_n + \dots$$

We arrange the terms in $\Delta(x^\mu r^m U_0)$ by the degree up to order k and group the remaining ones in a remainder. Precisely,

$$\begin{aligned}\Delta(x^\mu r^m U_0) &= \frac{U_0}{r} [m(m+1+2\mu_n) x^\mu r^{m-1} + \mu_n x^{\mu - \bar{n}} r^m + \\ &\quad + \mu_i(\mu_i - 1) x^{\mu - 2\bar{i}} r^{m+1} + c_{\sigma l}^{\mu m} x^\sigma r^l + w^{\mu m}(x, r)],\end{aligned}$$

with

$$c_{\sigma l}^{\mu m} \neq 0 \quad \text{only if} \quad |\mu| + m - 1 < |\sigma| + l \leq k,$$

and

$$w^{\mu m}(x, r) = r^m w_m^\mu(x) + m r^{m-1} w_{m-1}^\mu(x)$$

with w_m^μ and w_{m-1}^μ of class $C_x^{k,\alpha}$ with vanishing derivatives of all orders up to $k-m$ respectively $k-(m-1)$ at 0.

The monomials $c_{\sigma l}^{\mu m} x^\sigma r^l$ have strictly higher degree than the first terms and together with $w^{\mu m}$ can be thought as lower order terms. Notice that the coefficients $c_{\sigma l}^{\mu m}$ are linear combinations of polynomial coefficients at 0 of $\kappa(x)$, $d\kappa(x)$, ν^i , $d\nu^i$ from (6.1), and they vanish in the flat case $\Gamma = \{x_n = 0\}$.

Thus under the assumption $\|\Gamma\|_{C^{k+2,\alpha}} \leq \delta$ (achieved after a dilation), we may suppose that

$$(6.2) \quad |c_{\sigma l}^{\mu m}| \leq \delta, \quad \|w_m^\mu\|_{C^{k,\alpha}}, \|w_{m-1}^\mu\|_{C^{k,\alpha}} \leq \delta.$$

If

$$P = a_{\mu m} x^\mu r^m \quad \text{is a polynomial of degree } k+1,$$

then

$$\Delta(U_0 P) = \frac{U_0}{r} (A_{\sigma l} x^\sigma r^l + w(x, r)), \quad |\sigma| + l \leq k,$$

with

$$(6.3) \quad \begin{aligned}A_{\sigma l} &= (l+1)(l+2+2\sigma_n) a_{\sigma, l+1} + (\sigma_n + 1) a_{\sigma + \bar{n}, l} + \\ &\quad + (\sigma_i + 1)(\sigma_i + 2) a_{\sigma + 2\bar{i}, l-1} + c_{\sigma l}^{\mu m} a_{\mu m},\end{aligned}$$

and

$$w(x, r) = \sum_{m=0}^k r^m w_m(x),$$

with $w_m \in C_x^{k,\alpha}$ and derivatives up to order $k-m$ vanishing at zero. Again, under the assumption $\|\Gamma\|_{C^{k+2,\alpha}} \leq \delta$, we have

$$\|w_m\|_{C^{k,\alpha}} \leq \delta \max |a_{\mu\gamma}|.$$

From (6.3) we see that $a_{\sigma,l+1}$ (whose coefficient is different than 0) can be expressed in terms of $A_{\sigma l}$ and a linear combination of $a_{\mu m}$ with $\mu + m < |\sigma| + l + 1$ plus a linear combination of $a_{\mu m}$ with $\mu + m = |\sigma| + l + 1$ and $m < l + 1$. This shows that the coefficients $a_{\mu m}$ are uniquely determined from the linear system (6.3) once $A_{\sigma l}$ and $a_{\mu 0}$ are given.

Definition 6.1. We say that P is approximating for the equation (5.1) if $A_{\sigma l}$ coincide with the coefficients of R .

To obtain the proof of Proposition 5.1 in the general case, is now enough to obtain the same improvement of flatness as in Lemma 5.2, with the approximating polynomials defined above

Indeed, assume that (after a dilation)

$$\|\Gamma\|_{C^{k+2,\alpha}} \leq \delta, \quad |R| \leq \delta, \quad |F| \leq \delta r^{-\frac{1}{2}} |X|^\alpha.$$

Since P is approximating, arguing as in the case $k = 0$ we have

$$u - U_0 P =: \lambda^{k+\frac{3}{2}+\alpha} \tilde{u}\left(\frac{X}{\lambda}\right),$$

and

$$(6.4) \quad \Delta \tilde{u}\left(\frac{X}{\lambda}\right) = \lambda^{\frac{1}{2}-k-\alpha} \left(F(X) - \frac{U_0}{r} w(x, r) \right) =: \tilde{F}\left(\frac{X}{\lambda}\right).$$

Using that $\lambda^{m-k} w_m(\lambda x)$ has bounded $C_x^{k,\alpha}$ norm in B_1 together with the hypothesis on F , we obtain

$$\|\tilde{F}(X)\|_{L^\infty(B_1)} \leq \delta.$$

Now the proof is the same as Lemma 5.2. The only difference is that the approximating polynomial Q has degree $k+1$ and satisfies (see (6.3))

$$(l+1)(l+2+2\sigma_n) q_{\sigma,l+1} + (\sigma_n+1) q_{\sigma+\bar{n},l} + (\sigma_i+1)(\sigma_i+2) q_{\sigma+2\bar{i},l-1} = 0,$$

with bounded $q_{\mu m}$. Then we need to modify Q into \bar{Q} such that $\bar{Q}(x/\lambda, r/\lambda)$ is approximating for $R \equiv 0$. Thus its coefficients solve the system (6.3) with $A_{\sigma l} = 0$ and rescaled $c_{\sigma l}^{\mu m}$, i.e.

$$(6.5) \quad (l+1)(l+2+2\sigma_n) \bar{q}_{\sigma,l+1} + (\sigma_n+1) \bar{q}_{\sigma+\bar{n},l} + (\sigma_i+1)(\sigma_i+2) \bar{q}_{\sigma+2\bar{i},l-1} + \bar{c}_{\sigma l}^{\mu m} \bar{q}_{\mu m} = 0,$$

with

$$\bar{c}_{\sigma l}^{\mu m} := \lambda^{|\sigma|+l+1-|\mu|-m} c_{\sigma l}^{\mu m}, \quad \text{hence} \quad |\bar{c}_{\sigma l}^{\mu m}| \leq |c_{\sigma l}^{\mu m}| \leq \delta.$$

After subtracting the last 2 equalities we see that the coefficients of $Q - \bar{Q}$ solve the linear system (6.5) with right hand side $A_{\sigma l} = \bar{c}_{\sigma l}^{\mu m} q_{\mu m}$, hence $|A_{\sigma l}| \leq C\delta$. Thus by choosing $\bar{q}_{\mu 0} = q_{\mu 0}$ we can solve uniquely for \bar{Q} and find

$$\|\bar{Q} - Q\| \leq C\delta.$$

This concludes the proof. \square

Now we sketch the proof of Theorem 3.3 which follows from Proposition 5.1.

Proof of Theorem 3.3. We assume that

$$\Delta u = F \quad \text{in } B_1 \setminus \mathcal{P}, \quad F \in C^{k,\alpha}(B_1),$$

and u vanishes continuously on \mathcal{P} , and $\Gamma = \partial_{\mathbb{R}^n} \mathcal{P} \in C^{k+2}$. We decompose $u = u_{ev} + u_{od}$ in the even and odd part which solve the Laplace equation with right hand side F_{ev} respectively F_{od} .

We have the following expansions

$$u_{od} = x_{n+1} (P_{od}(x, x_{n+1}) + O(|X|^{k+1})), \quad \deg P_{od} = k,$$

for some polynomial P_{od} , even in x_{n+1} . For the even part we can write

$$\bar{u}(X) := u_{ev}(X) - x_{n+1}^2 T_0(x, x_{n+1}), \quad \deg T = k,$$

for some appropriate even polynomial T_0 , such that

$$\Delta \bar{u} = \bar{F} \quad \text{with} \quad \bar{F} = O(|X|^{k+\alpha}).$$

We can apply for \bar{u} Proposition 5.1 at the origin and obtain

$$\bar{u} = U_0(P_0(x, r) + O(|X|^{k+1+\alpha})).$$

In conclusion

$$u = U_0 P_0 + x_{n+1} P_{od} + x_{n+1}^2 T_0 + O(U_0 |X|^{k+1+\alpha}).$$

Writing this at all points on Γ and using the arguments of Remark 5.3 we obtain

$$u = U_0 \left(\sum_{m=0}^{k+1} a_m(x) r^m + O(r^{k+1+\alpha}) \right) + x_{n+1} \left(\sum_{m=0}^k b_m(x) x_{n+1}^m \right),$$

for functions

$$a_m \in C^{k+1-m,\alpha}, \quad b_m \in C^{k-m,\alpha+\frac{1}{2}}.$$

\square

We conclude this section with the estimates for the derivatives of harmonic functions in slit domains.

Proposition 6.2. *Assume $\Gamma \in C^{k+2,\alpha}$, $k \geq 0$, and $u \in C(B_1)$, even, satisfies*

$$\Delta u = 0 \quad \text{in } B_1 \setminus \mathcal{P}, \quad u = 0 \quad \text{on } \mathcal{P}.$$

If $|\mu| \leq k + 2$, then

$$D_x^\mu u = \frac{U_0}{r^{2|\mu|-1}} (P_0^\mu(x, r) + O(|X|^{k+|\mu|+\alpha})),$$

with P_0^μ having monomials with degrees between $|\mu| - 1$ and $k + |\mu|$.

Moreover, $U_0 r^{1-2|\mu|} P_0^\mu$ is obtained by differentiating formally $D_x^\mu(U_0 P_0)$ at 0.

Indeed, as we discussed in Lemma 5.5 in the case $k = 0$, these estimates follow from the proof of Proposition 3.1. From (6.4) we see that \tilde{F} is pointwise $C^{k+1,\alpha}$ on the set $\mathcal{P} \cap \mathcal{C}$. Hence if $|\mu| \leq k + 2$,

$$|D_x^\mu \tilde{u}| \leq C U_0 \quad \text{in } \mathcal{C}',$$

which gives the conclusion in the cone $\{r \geq |x'|\}$. However, this expansion is valid around each such cone centered on Γ . Then as in Remark 5.6 we can show that the conclusion holds in fact in a whole neighborhood of 0.

7. PROOF OF THEOREM 4.1

In this section we prove Theorem 4.1. We assume throughout that

$$\Gamma \in C^{k+2,\alpha}, \quad \|\Gamma\|_{C^{k+2,\alpha}} \leq \delta,$$

$$\Delta u = 0 \quad \text{in } B_1 \setminus \mathcal{P}, \quad u \text{ is even and } \frac{1}{2}U_0 \leq u \leq 2U_0,$$

and $w \in C(B_1)$, even, $\|w\|_{L^\infty} \leq 1$ solves

$$(7.1) \quad \begin{cases} \Delta(u_n w) = 0 & \text{in } B_1 \setminus \mathcal{P}, \\ w_\nu = 0 & \text{on } \Gamma \cap B_1. \end{cases}$$

We want to show that $w \in C_x^{k+2,\alpha}(0)$, that is we can find a polynomial $T_0(x, r)$, $\deg T_0 = k + 2$, such that

$$|w - T_0| \leq C|X|^{k+2+\alpha},$$

with C depending on n, k, α . Throughout this section we use $O(|X|^\beta)$ as a notation for functions that are bounded by $C|X|^\beta$ with C depending only on n, k, α .

The proof follows the lines of the proof of Proposition 5.1, however it is more technical since it involves the singular weight u_n . This time we do not approximate directly w by a polynomial of degree $k + 2$, but rather by a sum between $rP(x, r)$, with $\deg P = k + 1$, and a $C_x^{k+2,\alpha}$ function of x with vanishing normal derivative on Γ . This function of x has also the property that it solves (7.1) with a controlled right hand side. A polynomial Q of degree $k + 2$ in x does not have these properties, and we need to adjust it in order to satisfy them. Next we construct such functions.

7.1. Definition of $E(Q)$. Let

$$y \rightarrow x = (y', g(y')) + y_n \nu, \quad \nu := \frac{(-\nabla g, 1)}{\sqrt{1 + |\nabla g|^2}}$$

be a change of coordinates from \mathbb{R}^n to \mathbb{R}^n which maps $\{y_n = 0\}$ to Γ and the lines in the y_n -direction into lines perpendicular to Γ . Since $g \in C^{k+2,\alpha}$ this change of coordinates is of class $C^{k+1,\alpha}$, and at least formally it is of class $C^{k+2,\alpha}$ pointwise on $\{y_n = 0\}$. Let

$$(7.2) \quad Q = Q(y') = q_\mu y^\mu, \quad |\mu| \leq k + 2, \quad q_\mu = 0 \quad \text{if } \mu_n \neq 0,$$

be a $k + 2$ polynomial in y' (hence it is constant in the y_n -direction).

We work with such polynomials Q viewed as functions of the x -variable. As a function of x , Q is only a $C^{k+1,\alpha}$ function. However, we show below that on Γ , Q is pointwise $C^{k+2,\alpha}$, that is it can be approximated by a polynomial of degree $k + 2$ in x with an error of order $k + 2 + \alpha$.

Claim: $Q \in C_x^{k+2,\alpha}(\Gamma)$.

Proof: It suffices to show that each coordinate function y_i , $1 \leq i \leq n$, is in $C_x^{k+2,\alpha}(\Gamma)$. We show this at the origin by comparing the corresponding coordinate functions y_i (as functions of x) for Γ and Γ_t its tangent $k + 2$ polynomial at the origin. The coordinate functions y_i differ in B_ρ by $C\rho^{k+2+\alpha}$ since Γ and Γ_t differ

by $C\rho^{k+2+\alpha}$ and the normals ν_Γ and ν_{Γ_t} differ by $C\rho^{k+1+\alpha}$. Clearly, in the case of Γ_t , y_i is $C_x^{k+2,\alpha}$ at the origin. Thus the same holds for Γ . \square

Now we extend (regularize) Q away from Γ without changing its $k+2$ tangent polynomials on Γ . The extension $E(Q)$ has the property that it is of class $C^{k+2,\alpha}$ at all x 's and on Γ it coincides with Q up to order $k+2$. The existence of $E(Q)$ follows from Whitney's extension theorem (see for example [F]). For completeness, we present its simple proof for our case in the Appendix. Precisely, we have the following theorem.

Theorem 7.1 (Whitney Extension Theorem). *There exists $E(Q)$ such that*

$$D_x^\mu E(Q) = D_x^\mu Q \quad \text{on } \Gamma, \text{ for all } |\mu| \leq k+2$$

and

$$\|E(Q)\|_{C_x^{k+2,\alpha}(B_1)} \leq C\|Q\|_{C_x^{k+2,\alpha}(\Gamma)}.$$

Moreover $E(Q)$ is linear in Q , and if Q is given by (7.2), then

$$E(Q) = \tilde{q}_\mu x^\mu + O(|x|^{k+2+\alpha})$$

with

$$\tilde{q}_\mu = q_\mu + \tilde{c}_\mu^\sigma q_\sigma, \quad \tilde{c}_\mu^\sigma \neq 0 \quad \text{only if } |\sigma| < |\mu|.$$

The last claim follows from the fact that $E(Q)$ and Q have the same tangent polynomial at 0 and for $1 \leq i \leq n-1$ we write y_i as a polynomial of degree $k+2$ in x plus an error $O(|x|^{k+2+\alpha})$. The first order in each expansion is

$$y_i = x_i + \text{lower order terms},$$

and \tilde{c}_μ^σ depend on the derivatives up to order $k+2$ of g at the origin.

In the proof of Theorem 4.1, we approximate w at the origin by the sum of $E(Q)$ for some Q as in (7.2) and a function in $C_{xr}^{k+2,\alpha}(0)$.

7.2. Properties of $E(Q)$. First we notice that, since Q is constant on perpendicular lines to Γ then $Q_\nu = 0$ on Γ . Thus

$$E(Q)_\nu = 0 \quad \text{on } \Gamma.$$

In the next lemma we estimate $\Delta(u_e E(Q))$ for some unit vector e .

Lemma 7.2. *Let e be a unit vector and let u_e have the following expansion at 0,*

$$u_e = \frac{U_0}{r}(P_0^e + O(|X|^{k+1+\alpha})), \quad \deg P_0^e = k+1.$$

Then,

$$\Delta(u_e E(Q)) = \frac{U_0}{r}(R + O(|X|^{k+\alpha})) \quad \text{in } B_1 \setminus \mathcal{P},$$

with R a polynomial of degree k in (x, r) and

$$R = A_{\sigma l} x^\sigma r^l, \quad |\sigma| + l \leq k$$

with

$$A_{\sigma l} = \begin{cases} c_{\sigma l}^\mu q_\mu, & \text{if } (\sigma_n, l) \neq (0, 0), \\ P_0^e(0)(\sigma_i + 1)(\sigma_i + 2)q_{\sigma+2\bar{i}} + c_{\sigma l}^\mu q_\mu & \text{if } (\sigma_n, l) = (0, 0), \end{cases}$$

and

$$c_{\sigma l}^\mu \neq 0 \quad \text{only if } |\mu| \leq |\sigma| + l + 1.$$

Proof. Since $\Delta u_e = 0$ we have

$$\Delta(u_e E(Q)) = u_e \Delta E(Q) + 2\nabla u_e \cdot \nabla E(Q).$$

From Theorem 7.1 we know that $\Delta E(Q)$ is pointwise $C_x^{k,\alpha}$ at the origin and its expansion is obtained by formally differentiating the expansion of $E(Q)$ at the origin. Next we estimate the second term by making use that $\nabla E(Q)$ is almost parallel to Γ .

We claim that

$$\nabla E(Q) = \nabla Q + |d|^{k+\alpha} \xi + |d|^{k+\alpha+1} \eta$$

for two bounded vectors $\xi, \eta \in \mathbb{R}^n$ with $\xi \cdot \nu = 0$.

Assume for simplicity that x is a point on the e_n -axis. Then, since $E(Q) \in C_x^{k+2,\alpha}$ we find

$$\nabla E(Q)(x) = \sum_{m=0}^{k+1} \frac{1}{m!} x_n^m \nabla \partial_n^m Q(0) + O(|x_n|^{k+1+\alpha}).$$

Since $Q \in C_x^{k+1,\alpha}$ we see by Taylor expansion that

$$\sum_{m=0}^{k+1} \frac{1}{m!} x_n^m \nabla \partial_n^m Q(0) = \nabla Q(x) + \xi |d|^{k+\alpha},$$

for some bounded vector ξ . Moreover, since Q is constant on perpendicular lines to Γ , $\nabla Q(x) \cdot e_n = 0$ and $\partial_n^l Q(0) = 0$ for all $l \leq k+2$. Thus, the formula above gives $\xi \cdot e_n = 0$ and our claim is proved.

From Proposition 6.2

$$u_{ei} = \frac{U_0}{r^2} \left[\left(\frac{1}{2} - \frac{d}{r} \right) \nu^i P_0^e + r \partial_i P_0^e + d \nu^i \partial_r P_0^e + O(|X|^{k+1+\alpha}) \right].$$

Thus at a point X in the 2D plane perpendicular to Γ at 0, i.e. $\{x' = 0\}$, we have $\xi \cdot e_n = 0$, $\nabla Q \cdot e_n = 0$, $\nu^i = 0$ for $i \neq n$, hence

$$|\xi \cdot \nabla u_e| \leq C \frac{U_0}{r}, \quad |\eta \cdot \nabla u_e| \leq C \frac{U_0}{r^2}.$$

and

$$\nabla u_e \cdot \nabla Q = \frac{U_0}{r} [\partial_i P_0^e \partial_i Q + O(r^{k+\alpha})].$$

This means that at an arbitrary point X we find

$$\nabla u_e \cdot \nabla E(Q) = \frac{U_0}{r} [\partial_x P_Z^e \cdot \nabla Q + O(r^{k+\alpha})],$$

where P_Z^e is the $k+1$ order polynomial in the expansion of u_e at $Z \in \Gamma$, projection of X onto Γ . Also, for a polynomial $P(x, r)$, $\partial_x P(x, r)$ denotes the gradient with respect to x with r thought as independent of x .

As in Remark 5.3 we may replace $\partial_x P_Z^e$ with $\partial_x P_0^e$ and create an error of order $O(|Z|^{k+\alpha})$. In conclusion

$$\nabla u_e \cdot \nabla E(Q) = \frac{U_0}{r} (\partial_x P_0^e \cdot \nabla Q + O(|X|^{k+\alpha})),$$

and

$$\Delta(u_e E(Q)) = \frac{U_0}{r} [P_0^e \Delta E(Q) + \partial_x P_0^e \cdot \nabla Q + O(|X|^{k+\alpha})],$$

and the conclusion follows by using the expansions for $E(Q)$ and Q at the origin. \square

We remark that the coefficients $c_{\sigma l}^\mu$ depend on the coefficients of P_0^ε and \tilde{c}_μ^σ .

7.3. Compactness of solutions to (7.1). Let P_0 be the approximating polynomial for u at 0 (given by Theorem 3.1). From now we assume, after multiplying u by a constant (recall that $\frac{1}{2}U_0 \leq u \leq 2U_0$), that $P_0(0) = 1$ thus

$$(7.3) \quad u = U_0(1 + O(|X|)),$$

and then we find (see (3.4))

$$(7.4) \quad u_n = \frac{U_0}{r} \left(\frac{1}{2} + O(|X|) \right).$$

Notice that the rescalings

$$\tilde{u}(X) := \lambda^{-1/2} u(\lambda X)$$

satisfy the same properties and as $\lambda \rightarrow 0$ and

$$\tilde{u} \rightarrow U_0, \quad \tilde{u}_n \rightarrow (U_0)_n, \quad \tilde{\Gamma} \rightarrow L := \{x_n = 0, x_{n+1} = 0\}.$$

Next we prove that w is uniformly Hölder continuous. We prove this under more relaxed hypotheses on w .

Lemma 7.3. *Let u be as above and let w satisfy*

$$\begin{cases} |\Delta(u_n w)| \leq \frac{U_0}{r} & \text{in } B_1 \setminus \mathcal{P} \\ |w_\nu| \leq 1 & \text{on } \Gamma, \quad \|w\|_{L^\infty(B_1)} \leq 1. \end{cases}$$

Then, $w \in C^\beta$ and $\|w\|_{C^\beta(B_{1/2})} \leq C$, for some β small, universal.

Proof. The fact that $w \in C^\beta$ away from Γ is obvious. We only need to show that the oscillation of w as we approach Γ decreases at a geometric rate.

The rescalings

$$\tilde{w}(x) = w(\lambda x)$$

satisfy in B_1

$$\begin{cases} |\Delta(\tilde{u}_n \tilde{w})| \leq \lambda^2 \frac{\tilde{U}_0}{r} & \text{in } B_1 \setminus \mathcal{P} \\ |\tilde{w}_\nu| \leq \lambda & \text{on } \Gamma, \quad \|\tilde{w}\|_{L^\infty(B_1)} \leq 1. \end{cases}$$

Thus by scaling, it suffices to show that if

$$\begin{cases} |\Delta(u_n w)| \leq \delta_0 \frac{U_0}{r} & \text{in } B_1 \setminus \mathcal{P} \\ |w_\nu| \leq \delta_0 & \text{on } \Gamma, \quad \|w\|_{L^\infty(B_1)} \leq 1, \end{cases}$$

with $|u - U_0| \leq \delta_0$, $\|g\|_{C^{2,\alpha}} \leq \delta_0$, then

$$\text{osc}_{B_{\delta_0}} w \leq 2 - \delta_0,$$

for some δ_0 universal.

Assume $w(\frac{1}{2}e_n) \geq 0$. We construct a lower barrier for w defined as

$$v := -1 + \delta_1 \left(\frac{1}{4} + E(Q) + \frac{U_0}{u_n} (1 + Mr) \right)$$

with $Q(y') = -|y'|^2$, and some δ_1 small, and M large to be made precise later. From Lemma 7.2

$$\Delta(u_n E(Q)) \geq -C \frac{U_0}{r},$$

and we first choose M large such that (see (5.3))

$$\Delta(U_0(1 + Mr)) \geq (cM - C)\frac{U_0}{r} \geq 2C\frac{U_0}{r}.$$

Notice that on Γ , (see (7.4))

$$\partial_\nu \left(E(Q) + \frac{U_0}{u_n}(1 + Mr) \right) \geq c.$$

We compare w and v in the cylindrical region $B_{3/4} \cap \{r < c\}$. We have

$$v < -1 \leq w \quad \text{on } \partial B_{3/4} \cap \{r < c\},$$

provided that we take c sufficiently small.

Since $w(\frac{1}{2}e_n) \geq 0$, by Harnack inequality (and boundary Harnack) for $u_n w$ we obtain that $w \geq -1 + c_0$ on $B_{3/4} \cap \{r = c\}$. If we choose δ_1 small enough, then we have

$$v \leq w \quad \text{on } B_{3/4} \cap \{r = c\}.$$

If $\delta_0 \ll \delta_1$ we have $\Delta(u_n w) \leq \Delta(u_n v)$ in $(B_{3/4} \cap \{r < c\}) \setminus \mathcal{P}$. Then $v \leq w$ by the maximum principle and the conclusion easily follows since $v \geq -1 + \delta_1/8$ in a neighborhood of 0. Indeed, the minimum of $w - v$ cannot occur on Γ because of the free boundary condition, cannot occur on \mathcal{P} because of Hopf lemma and cannot occur in the interior because of the classical maximum principle. \square

Lemma 7.4 (Compactness). *Let $u_k(x)$ be a sequence of harmonic functions in $B_1 \setminus \mathcal{P}_k$, vanishing on \mathcal{P}_k , with*

$$u_k = U_0(1 + \delta_k O(|X|)) \quad \|\Gamma_k\|_{C^{2,\alpha}} \leq \delta_k,$$

for a sequence $\delta_k \rightarrow 0$. Let w_k satisfy

$$\begin{cases} |\Delta((\partial_n u_k)w_k)| \leq \delta_k \frac{U_0}{r}, & \text{on } B_1 \setminus \mathcal{P} \\ |\partial_\nu w_k| \leq \delta_k & \text{on } \Gamma, \quad \|w_k\|_{L^\infty(B_2)} \leq 1. \end{cases}$$

Then there is a subsequence of w_k that converges uniformly on compact sets to \bar{w} that satisfies the limiting equation in the flat case i.e. $\bar{\mathcal{P}} = \{x_n < 0\}$, $\bar{\Gamma} = \{x_n = 0\}$,

$$\begin{cases} \Delta((U_0)_n \bar{w}) = 0 & \text{in } B_1 \setminus \bar{\mathcal{P}} \\ \partial_\nu \bar{w} = 0 & \text{on } \bar{\Gamma}. \end{cases}$$

Remark 7.5. The free boundary condition for \bar{w} is understood in the viscosity sense defined in [DS1], i.e. \bar{w} cannot be touched on L say at 0 by below by a function of the form

$$b - a_1|y' - y'_0|^2 + a_2 r \quad \text{with } a_1 > 0 \text{ and } a_2 > 0,$$

for some constants b_0, a_1, a_2 and some vector y'_0 .

Proof. The fact that $w_k \rightarrow \bar{w}$ (up to a subsequence) on compact subsets of B_1 follows from Lemma 7.3. Also, from our assumptions $u_k \rightarrow U_0, \Gamma_k \rightarrow \bar{\Gamma}$.

Clearly, $(U_0)_n \bar{w}$ is harmonic in the interior. It remains to check the condition on $\bar{\Gamma}$, that is we cannot touch \bar{w} by below with a function as above. Otherwise, we can also touch by below \bar{w} strictly in a neighborhood of 0 with the function

$$b_0 - 2a_1|y' - y'_1|^2 + \frac{a_2}{2}r + Mr^2 \quad \text{with } M \gg 2a_1,$$

for some b_0 and y'_1 . Since $w_k \rightarrow \bar{w}$ uniformly, then we can touch w_k by

$$v_k := b_k - E(2a_1|y' - y'_1|^2) + \frac{U_0}{u_n}(\frac{a_2}{4} + \frac{M}{2}r),$$

for some constant b_k . As in the proof of Lemma 7.3, v_k is a strict subsolution to our Neumann problem for all k large and we reach a contradiction. \square

7.4. Proof of Theorem 4.1. We argue similarly to the proof of Proposition 5.1 and we approximate w inductively in sequence of balls B_{ρ^m} . However, in this case we do not use directly polynomials of degree $k+2$ in (x, r) , but rather functions which are pointwise $C_{xr}^{k+2, \alpha}(0)$ and approximate better the Neumann problem. Precisely we use functions of the type

$$W_{Q,P} := E(Q) + \frac{U_0}{u_n}P$$

with Q a polynomial of degree $k+2$ in y as in (7.2) i.e.

$$Q = q_\mu y^\mu, \quad |\mu| \leq k+2, \quad q_\mu = 0 \quad \text{if } \mu_n \neq 0,$$

and P a polynomial of degree $k+1$ in (x, r) ,

$$P = a_{\mu m} x^\mu r^m, \quad |\mu| + m \leq k+1.$$

By Theorem 3.1 we have

$$(7.5) \quad \frac{U_0}{u_n} = r \left(P_*(x, r) + O(|X|^{k+1+\alpha}) \right), \quad \deg P_* = k+1,$$

and also by (7.3), $P_*(0, 0) = 2$.

We say that a pair (Q, P) is approximating for the Neumann problem (7.1) if:

- (i) P_*P vanishes of order $k+1$ on Γ ;
- (ii) P is approximating as in Definition 6.1 for $-R$ from Lemma 7.2. That is, the coefficients of P satisfy the system (6.3) with left hand side $-A_{\sigma l}$ where $A_{\sigma l}$ is given in Lemma 7.2 (with $e = e_n$).

Condition (i) says that on Γ

$$(7.6) \quad \partial_\nu W_{Q,P} = O(|X|^{k+1+\alpha}),$$

and condition (ii), in view of Lemma 7.2, gives

$$(7.7) \quad \Delta(u_n W_{Q,P}) = O\left(\frac{U_0}{r} |X|^{k+\alpha}\right).$$

We write below the two conditions above in terms of the coefficients of Q and P . For convenience, we relabel the coefficients as

$$b_{\mu,0} := q_\mu, \quad b_{\mu,m+1} := a_{\mu m} \quad \text{for } m \geq 0.$$

Precisely (i) says that by taking $r = 0$ and $x_n = g(x')$ in P_*P , then P_*P vanishes of order $k+1$ (in $|x'|$) at the origin. Hence, by looking at the coefficient of $x^{\mu'}$ we find

$$(7.8) \quad b_{(\sigma',0),1} = \hat{c}_{\sigma'}^\mu b_{\mu,1}, \quad \hat{c}_{\sigma'}^\mu \neq 0 \quad \text{only if } |\mu| < |\sigma'|,$$

with $\hat{c}_{\sigma'}^\mu$ depending on the derivatives of g and the coefficients of P_* . Thus $b_{\mu,1}$ are determined uniquely from the linear system (7.8) once $b_{\mu,1}$ with $\mu_n \neq 0$ have been fixed.

Property (ii) can be written as

$$\begin{aligned}
 (7.9) \quad 0 &= (l+1)(l+2+2\sigma_n)b_{\sigma,l+2} + (\sigma_n+1)b_{\sigma+\bar{n},l+1} + \\
 &\quad + (\sigma_i+1)(\sigma_i+2)b_{\sigma+2\bar{i},l} + \bar{c}_{\sigma l}^{\mu m} b_{\mu,m}, \quad \text{if } (\sigma_n, l) \neq (0, 0); \\
 0 &= (l+1)(l+2+2\sigma_n)b_{\sigma,l+2} + (\sigma_n+1)b_{\sigma+\bar{n},l+1} + \\
 &\quad + \frac{1}{2}(\sigma_i+1)(\sigma_i+2)b_{\sigma+2\bar{i},l} + \bar{c}_{\sigma l}^{\mu m} b_{\mu,m}, \quad \text{if } (\sigma_n, l) = (0, 0),
 \end{aligned}$$

with

$$\bar{c}_{\sigma l}^{\mu m} \neq 0 \quad \text{only if } |\mu| + m < \sigma + l + 2.$$

In this system $b_{\sigma,l+2}$ is determined by a linear combination of $b_{\mu,m}$'s with $|\mu| + m \leq |\sigma| + l + 2$ and in case of equality, of $b_{\mu,m}$'s with $m < l + 2$. Thus the coefficients $b_{\mu,m}$ are determined uniquely from this system once $b_{\mu,0}$ and $b_{\mu,1}$ have been fixed.

In conclusion all coefficients $b_{\mu,m}$ are determined uniquely from the two linear systems (7.8)-(7.9) once $b_{\mu,0}$ and $b_{\mu,1}$ with $\sigma_n \neq 0$, are given. Notice that, by definition, we always take $b_{\mu,0} = 0$ if $\mu_n \neq 0$.

Now we proceed with the proof of the theorem. After an initial dilation, we may suppose that we are close enough to the linear case, that is (see (7.3)),

$$\|g\|_{C^{k+2,\alpha}(B_1)} \leq \delta, \quad u_n = \frac{U_0}{r} (P_0^n(x, r) + \delta O(|X|^{k+1+\alpha})),$$

for some polynomial P_0^n of degree $k+1$ with

$$\|P_0^n - \frac{1}{2}\| \leq \delta.$$

Using this in Lemma 7.2 and in (7.5), we find that

$$|\bar{c}_{\sigma l}^{\mu m}| \leq C\delta, \quad |\bar{c}_{\sigma l}^{\mu}| \leq C\delta,$$

and (7.6)-(7.7) hold with the right hand side multiplied by δ , that is

$$\begin{aligned}
 \partial_\nu W_{Q,P} &= \delta O(|X|^{k+1+\alpha}), \\
 \Delta(u_n W_{Q,P}) &= \delta O\left(\frac{U_0}{r} |X|^{k+\alpha}\right).
 \end{aligned}$$

It suffices to show that if w satisfies (7.1) and

$$|w - W_{Q,P}| \leq \lambda^{k+2+\alpha} \quad \text{in } B_\lambda, \quad \lambda \leq 1,$$

for some approximating pair (Q, P) with $\|Q\|, \|P\| \leq 1$, then

$$|w - W_{\bar{Q}, \bar{P}}| \leq (\rho\lambda)^{k+2+\alpha} \quad \text{in } B_\lambda, \quad \|(\bar{Q} + r\bar{P}) - (Q + rP)\|_{L^\infty(B_\lambda)} \leq C\lambda^{k+2+\alpha},$$

for some approximating pair (\bar{Q}, \bar{P}) . Then the theorem follows by applying this result inductively by starting with the initial approximating pair $(0, 0)$ in B_1 .

We prove the claim above similarly as in Lemma 5.2. We write

$$w = W_{Q,P} + \lambda^{k+2+\alpha} \tilde{w}(X/\lambda).$$

Then $\|\tilde{w}\|_{L^\infty(B_1)} \leq 1$ and

$$\Delta(\tilde{u}_n \tilde{w}) = \lambda^{\frac{1}{2}-k-\alpha} \Delta(u_n W_{P,Q}), \quad \tilde{w}_\nu = \lambda^{-(k+1+\alpha)} \partial_\nu(W_{Q,P}).$$

Using (7.6)-(7.7) we have that in B_1 ,

$$|\Delta(\tilde{u}_n \tilde{w})| \leq C\delta \frac{\tilde{U}_0}{r}, \quad |\tilde{w}_\nu| \leq \delta \quad \text{on } \tilde{\Gamma}.$$

Thus by the compactness Lemma 7.4 and Theorem 4.6

$$|\tilde{w} - \tilde{Q}(x') - r\tilde{P}| \leq \frac{1}{4}\rho^{k+2+\alpha} + C\rho^{k+3} \leq \frac{1}{2}\rho^{k+2+\alpha} \quad \text{in } B_\rho,$$

and (\tilde{Q}, \tilde{P}) solves the system (7.6)-(7.7) with vanishing constants $\tilde{c}_{\sigma'}^\mu, \tilde{c}_{\sigma l}^{\mu m}$. As before, we can modify (\tilde{Q}, \tilde{P}) above into (\bar{Q}, \bar{P}) such that

$$\|(\tilde{Q} + r\tilde{P}) - (\bar{Q} + r\bar{P})\| \leq C\delta, \quad \text{and } (\bar{Q}, \bar{P})(X/\lambda) \text{ is approximating.}$$

By taking δ sufficiently small we obtain

$$\left| \tilde{w} - \left(E(\bar{Q}) + \frac{\tilde{U}_0}{u_n} \bar{P} \right) \right| \leq \rho^{k+2} \quad \text{in } B_\rho,$$

which gives the desired claim. \square

8. APPENDIX

We now prove our estimates for the constant coefficients case (see Section 4.3 for the statements.)

Proof of Theorem 4.5. The function u is uniformly Hölder continuous on compact sets of B_1 . Moreover, since the equation is invariant after differentiating in the x' direction we find

$$\|D_{x'}^\mu u\|_{C^\beta(B_{1/2})} \leq C(|\mu|), \quad \mu = (\mu_1, \dots, \mu_{n-1}, 0, 0),$$

i.e. u is C^∞ in the x' variable. We write the equation as

$$\Delta_{(x_n, x_{n+1})} u = -\Delta_{x'} u =: f(x),$$

and due to the invariance of the equation in the x' direction, f and u have the same regularity properties. In particular they depend in a C^∞ fashion on the x' variable.

We determine the behavior of u in the (x_n, x_{n+1}) variables by solving the Laplace equation above in each two dimensional plane $x' = \text{const}$. Using the complex change of variables $z \rightarrow z^2$ i.e.

$$\bar{u}(z) := u(z^2), \quad \bar{f}(z) := f(z^2) \quad z := x_n + ix_{n+1},$$

we find

$$\Delta \bar{u} = 4|z|^2 \bar{f},$$

and \bar{u} vanishes on $x_n = 0$. After an odd reflection with respect to x_n , we see that the equation above is satisfied for functions \bar{u} and \bar{f} which are even in x_{n+1} , odd in x_n and have the same regularity properties. This easily implies that \bar{u} and \bar{f} are C^∞ in z . Moreover, \bar{u} has a polynomial expansion at 0 of the type

$$\bar{u} = x_n (P(x_n^2, x_{n+1}^2) + O(|z|^{2k+2})), \quad \deg P = k.$$

We obtain the desired result by writing $P(x_n^2, x_{n+1}^2)$ as a polynomial of degree k in the variables $Re z^2 = x_n^2 - x_{n+1}^2$ and $|z|^2 = x_n^2 + x_{n+1}^2$, and then scaling back to u .

The claim that $U_0 P_0$ is harmonic in $B_1 \setminus \mathcal{P}$ follows from scaling. Indeed let $P_0 = \sum_{m=0}^k p_0^m(x, r)$ with each p_0^m a homogeneous polynomial of degree m . We

argue by induction on m . Clearly the statement is true for $m = 0$. Assume it is true for all $m \leq l < k$. Then,

$$(8.1) \quad v := u - U_0 \sum_{m=0}^l p_0^m = U_0(p_0^{l+1}(x, r) + o(|X|^{l+1}))$$

and the function v is harmonic. We rescale,

$$v_\lambda(X) = \frac{v(\lambda X)}{\lambda^{1/2+l+1}}$$

and obtain a sequence of harmonic functions which by (8.1) tend to $U_0 p_0^{l+1}$ as $\lambda \rightarrow 0$. Thus $U_0 p_0^{l+1}$ is harmonic as well. \square

Proof of Theorem 4.6. This was proved in [DS1], and its proof is similar to the proof above. We sketch below a slightly different proof that uses Theorem 3.1.

In Lemma 7.3 we already obtained uniform Hölder continuity of solutions w on compact sets. By the invariance of the equation in the x' direction we obtain that w depends in a C^∞ fashion in the x' variable. Using barriers similar to the ones in Lemma 7.4 one can easily obtain that

$$(8.2) \quad |w(X) - w(x', 0, 0)| \leq Cr \quad \text{in } B_{1/2}.$$

Then the function

$$\bar{w}(X) := w(X) - w(x', 0, 0),$$

satisfies

$$\Delta\left(\frac{U_0}{r}\bar{w}\right) = \frac{U_0}{r}f(x') \quad f(x') := -\Delta_{x'}w(x', 0, 0) \in C^\infty,$$

and, by (8.2), $v := (U_0/r)\bar{w}$ vanishes continuously on Γ . We may apply Theorem 3.1 to v and obtain

$$\frac{\bar{w}}{r} = \bar{P}(x, r) + O(|X|^k), \quad \deg \bar{P} = k - 1,$$

and the theorem is proved by writing

$$w(x', 0, 0) = \bar{Q}(x') + O(|x'|^{k+1}), \quad \deg \bar{Q} = k.$$

The claim that T solves the same problem as w now follows from scaling, as in the final part of the previous proof. \square

We conclude this appendix, with the proof of our needed version of the Whitney Extension Theorem.

Proof of Theorem 7.1. In our case, the extension $E(Q)$ can be constructed by a convolution type operator. Let ρ be a smooth function with support in $B_{1/2} \subset \mathbb{R}^n$, such that

$$\int_{\mathbb{R}^n} \rho \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} \rho x^\mu \, dx = 0 \quad \text{if } 1 \leq |\mu| \leq k + 2.$$

Then polynomials of degree $k + 2$ are left invariant after convolution with ρ :

$$P * \rho = P \quad \deg P = k + 2.$$

Define

$$E(Q)(x) := \int Q(y) \rho\left(\frac{x-y}{d}\right) d^{-n} dy,$$

where d denotes the distance from x to Γ .

We show that $E(Q)$ satisfies the required properties. It suffices to show that for indices μ with $|\mu| = k + 2$, say in $B_{\lambda/2}(\lambda e_n)$ with λ small, we have

$$[D^\mu E(Q)]_{C^\alpha(B_{\lambda/2}(\lambda e_n))} \leq C, \quad |D^\mu E(Q)(\lambda e_n) - D^\mu Q(0)| \leq C\lambda^\alpha.$$

Since Q is pointwise $C^{k+2,\alpha}$ at the origin we have

$$Q = P_0 + h, \quad \deg P_0 = k + 2, \quad \|h\|_{L^\infty(B_{2\lambda})} \leq C\lambda^{k+2+\alpha},$$

and

$$E(Q) = P_0 + E(h).$$

We need to show that in $B_{\lambda/2}(\lambda e_n)$,

$$[D^\mu E(h)]_{C^\alpha} \leq C, \quad |D^\mu E(h)| \leq C\lambda^\alpha, \quad |\mu| = k + 2.$$

Indeed, after a dilation of factor $1/\lambda$ we have for $x \in B_{1/2}(e_n)$

$$\overline{E(h)}(x) := E(h)(\lambda x) = \int_{B_2} h(\lambda y) \rho\left(\frac{x-y}{d}\right) d^{-n} dy,$$

with d the distance from x to Γ/λ . Notice that $d \in C_x^{k+2,\alpha}$ hence

$$\rho\left(\frac{x-y}{d}\right) d^{-n} \quad \text{has bounded } C_x^{k+2,\alpha} \text{ norm in } B_{1/2}(e_n).$$

Thus, by using the bound on h we find

$$\|D^\mu \overline{E(h)}\|_{C^\alpha(B_{1/2}(e_n))} \leq C\lambda^{k+2+\alpha},$$

which gives the desired result. \square

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